

UNIFORM CLOSE-TO-CONVEXITY RADIUS OF SECTIONS OF FUNCTIONS IN THE CLOSE-TO-CONVEX FAMILY

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ABSTRACT. The authors consider the class \mathcal{F} of normalized functions f analytic in the unit disk \mathbb{D} and satisfying the condition

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \quad z \in \mathbb{D}.$$

Recently, Ponnusamy et al. [12] have shown that $1/6$ is the uniform sharp bound for the radius of convexity of every section of each function in the class \mathcal{F} . They conjectured that $1/3$ is the uniform univalence radius of every section of $f \in \mathcal{F}$. In this paper, we solve this conjecture affirmatively.

1. PRELIMINARIES AND THE MAIN THEOREM

Let \mathcal{A} be the family of functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then the n -th *section/partial sum* of f , denoted by $s_n(f)(z)$, is defined to be the polynomial

$$s_n(f)(z) = z + \sum_{k=2}^n a_k z^k.$$

Let \mathcal{S} denote the class of functions in \mathcal{A} that are univalent in \mathbb{D} . Finally, let \mathcal{C} , \mathcal{S}^* and \mathcal{K} denote the usual geometric subclasses of functions in \mathcal{S} with convex, starlike and close-to-convex images, respectively (see [3]).

If $f \in \mathcal{S}$ is arbitrary, then the argument principle shows that the n -th section $s_n(f)(z)$ is univalent in each fixed compact disk $|z| \leq r$ (< 1) provided that n is sufficiently large. But then if we set $p_n(z) = r^{-1}s_n(f)(rz)$, then $p_n(z)$ is a polynomial that is univalent in the unit disk \mathbb{D} . Consequently, the set of univalent polynomials is dense with respect to the topology of locally uniformly in \mathcal{S} (see [3]). Suffridge [19] showed that even the subclass of polynomials with the highest coefficient $a_n = 1/n$ is dense in \mathcal{S} . Szegő [20] discovered that every section $s_n(f)$ is univalent in the disk $|z| < 1/4$ for all $f \in \mathcal{S}$ and for each $n \geq 2$. The radius $1/4$ is best possible as the Koebe function $k(z) = z/(1-z)^2$ shows. It is worth pointing out that the case $n = 3$ of Szegő's result is far from triviality.

File: Bpon6_Arxiv.tex, printed: 28-1-2014, 1.53

2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C55, 33C05.

Key words and phrases. Univalent, convex and close-to-convex functions; partial sums, convolution.

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In [15], Ruscheweyh established a stronger result by showing that the partial sums $s_n(f)(z)$ of f are indeed starlike in the disk $|z| < 1/4$ for functions f belonging not only to \mathcal{S} but also to the closed convex hull of \mathcal{S} . The following conjecture concerning the exact (largest) radius of univalence r_n of $f \in \mathcal{S}$ is still open (see [13] and [3, §8.2, p. 241–246]).

Conjecture A. *If $f \in \mathcal{S}$, then $s_n(f)$ is univalent in $|z| < 1 - \frac{3}{n} \log n$ for all $n \geq 5$.*

A surprising fact observed by Bshouty and Hengartner [2] is that the Koebe function is no more extremal for the above conjecture. On the other hand, this conjecture has been solved by using an important convolution theorem [16] for a number of geometric subclasses of \mathcal{S} , for example, the classes \mathcal{C} , \mathcal{S}^* and \mathcal{K} . Indeed, for $\phi(z) = z/(1-z)$, the sections $s_n(\phi)$ are known to be convex in $|z| < 1/4$ (see [5]). Moreover for the Koebe function $k(z) = z/(1-z)^2$, $s_n(k)$ is known to be starlike in $|z| < 1 - \frac{3}{n} \log n$ for $n \geq 5$ and hence, for the convex function $\phi(z) = z/(1-z)$, $s_n(\phi)$ is convex in $|z| < 1 - \frac{3}{n} \log n$ for $n \geq 5$. From a convolution theorem relating to the Pólya-Schoenberg conjecture proved by Ruscheweyh and Sheil-Small [16], it follows that all sections $s_n(f)$ are convex (resp. starlike, close-to-convex) in $|z| < 1/4$ whenever $f \in \mathcal{C}$ (resp. $f \in \mathcal{S}^*$ and $f \in \mathcal{K}$). Similarly, for $n \geq 5$, $s_n(f)$ is convex (resp. starlike, close-to-convex) in $|z| < 1 - \frac{3}{n} \log n$ whenever $f \in \mathcal{C}$ (resp. $f \in \mathcal{S}^*$ and $f \in \mathcal{K}$). An account of history of this and related information may be found in [3, §8.2, p. 241–246] and also in the nice survey article of Iliev [6]. For further interest on this topic, we refer to [4, 14, 17, 18] and recent articles [8, 9, 10, 11].

One of the important criteria for an analytic function f defined on a convex domain Ω , to be univalent in Ω is that $\operatorname{Re} f'(z) > 0$ on Ω (see [3, Theorem 2.16, p. 47]). The following definition is a consequence of it.

A function $f \in \mathcal{A}$ is said to be close-to-convex (with respect to g), denoted by $f \in \mathcal{K}_g$ if there exists a $g \in \mathcal{C}$ such that

$$(1) \quad \operatorname{Re} \left(e^{i\alpha} \frac{f'(z)}{g'(z)} \right) > 0, \quad z \in \mathbb{D},$$

for some real α with $|\alpha| < \pi/2$. More often, we consider \mathcal{K}_g (with $\alpha = 0$ in (1)) and $\mathcal{K} = \cup_{g \in \mathcal{C}} \mathcal{K}_g$. For functions in \mathcal{K}_g , we have the following result of Miki [7].

Theorem B. *Let $f \in \mathcal{K}_g$, where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then $s_n(f)$ is close-to-convex with respect to $s_n(g)$ in $|z| < 1/4$.*

In a recent paper [1], the present authors proved the following.

Theorem C. *Let $f \in \mathcal{K}$. Then every section $s_n(f)$ of f belongs to the class \mathcal{K} in the disk $|z| < 1/2$ for all $n \geq 46$.*

Choosing different convex functions g in [1], the authors have found the value $N(g) \in \mathbb{N}$ for $f \in \mathcal{K}_g$ such that $s_n(f) \in \mathcal{K}_g$ in a disk $|z| < r$ for all $n \geq N(g)$.

In [12], the authors consider the class \mathcal{F} of locally univalent functions f in \mathcal{A} satisfying the condition

$$(2) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \quad z \in \mathbb{D}.$$

The importance of this class is outlined in [12] and it was also remarked that the class \mathcal{F} has a special role on certain problems on the class of harmonic univalent mappings in \mathbb{D} (see [12] and the references therein). It is worth remarking that functions in \mathcal{F} are neither included in \mathcal{S}^* nor includes \mathcal{S}^* nor \mathcal{K} . It is well-known that $\mathcal{F} \subsetneq \mathcal{K} \subsetneq \mathcal{S}$ and hence, it is obvious from an earlier observation that for $f \in \mathcal{F}$, each $s_n(f)(z)$ is close-to-convex in $|z| < 1/4$. An interesting question is to determine the largest uniform disk with this property (see Conjecture 1 below). We now recall a recent result of Ponnusamy et al. [12].

Theorem D. *Every section of a function in the class \mathcal{F} is convex in the disk $|z| < 1/6$. The radius $1/6$ cannot be replaced by a greater one.*

In the same article the authors [12] observed that all sections functions of \mathcal{F} are close-to-convex in the disk $|z| < 1 - \frac{3}{n} \log n$ for $n \geq 5$. Consider

$$(3) \quad f_0(z) = \frac{z - z^2/2}{(1 - z)^2}.$$

We see that $f_0 \notin \mathcal{S}^*$, but $f_0 \in \mathcal{K}$. Also, f_0 is extremal for many extremal problems for the class \mathcal{F} . By investigating the second partial sum of $f_0 \in \mathcal{F}$, the authors conjectured the following.

Conjecture 1. *Every section $s_n(f)$ of $f \in \mathcal{F}$ is close-to-convex in the disk $|z| < 1/3$ and $1/3$ is sharp.*

In this article we solve this conjecture in the following form.

Theorem 1. *Every section $s_n(f)$ of $f \in \mathcal{F}$ satisfies $\operatorname{Re}(s_n(f)'(z)) > 0$ in the disk $|z| < 1/3$. In particular every section is close-to-convex in the disk $|z| < 1/3$. The radius $1/3$ cannot be replaced by a greater one.*

We remark that this result is much stronger than the original conjecture. The following lemma is useful in the proof of Theorem 1.

Lemma E. [12, Lemma 1] *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F}$, then the following estimates hold:*

- (a) $|a_n| \leq \frac{n+1}{2}$ for $n \geq 2$. Equality holds for $f_0(z)$ given by (3) or its rotation.
- (b) $\frac{1}{(1+r)^3} \leq |f'(z)| \leq \frac{1}{(1-r)^3}$ for $|z| = r < 1$. The bounds are sharp.
- (c) If $f(z) = s_n(z) + \sigma_n(z)$, with $\sigma_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$, then for $|z| = r < 1$ we have

$$|\sigma_n'(z)| \leq \frac{n(n+1)r^{n+2} - 2n(n+2)r^{n+1} + (n+1)(n+2)r^n}{2(1-r)^3}.$$

2. PROOF OF THEOREM 1

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{F}$. We shall prove that each partial sum $s_n(z) := s_n(f)(z)$ of f satisfies the condition $\operatorname{Re}(s'_n(z)) > 0$ in the disk $|z| < 1/3$ for all $n \geq 2$.

Let us first consider the second section $s_2(z) = z + a_2 z^2$ of f . A simple computation shows that

$$\operatorname{Re}(s'_2(z)) = 1 + \operatorname{Re}(2a_2 z).$$

From Lemma E(a), we have $|a_2| \leq 3/2$ and as a consequence of it we get

$$\operatorname{Re}(s'_2(z)) \geq 1 - 2|a_2||z| \geq 1 - 3|z|$$

which is positive provided $|z| < 1/3$. Thus, $s_2(z)$ is close-to-convex in the disk $|z| < 1/3$. To show that the constant $1/3$ is best possible, we consider the function $f_0 \in \mathcal{F}$ given in (3), namely,

$$f_0(z) = \frac{1}{2} \left[\frac{1}{(1-z)^2} - 1 \right] = z + \sum_{n=2}^{\infty} \left(\frac{n+1}{2} \right) z^n.$$

Let us denote by $s_{2,0}(z)$, the second partial sum $s_2(f_0)(z)$ of $f_0(z)$ so that $s_{2,0}(z) = z + (3/2)z^2$. Then we get $s'_{2,0}(z) = 1 + 3z$, which vanishes at $z = -1/3$. Thus the constant $1/3$ is best possible.

Next, let us consider the case $n = 3$. Each $f \in \mathcal{F}$ satisfies the analytic condition (2) and so we can write

$$(4) \quad 1 + \frac{2zf''(z)}{3f'(z)} = p(z),$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in \mathbb{D} and $\operatorname{Re} p(z) > 0$ in \mathbb{D} . From Carathéodory Lemma [3, p. 41] we get $|p_n| \leq 2$ for all $n \geq 2$. If we rewrite (4) in power series form, then

$$1 + \frac{2z(2a_2 + 6a_3 z + 12a_4 z^2 + \dots)}{3(1 + 2a_2 z + 3a_3 z^2 + \dots)} = 1 + p_1 z + p_2 z^2 + \dots.$$

Now comparing the coefficients of z and z^2 on both sides yields the relations

$$p_1 = \frac{4}{3}a_2 \quad \text{and} \quad p_2 = \frac{4}{3}(3a_3 - 2a_2^2).$$

As $|p_1| \leq 2$ and $|p_2| \leq 2$, we may rewrite the last two relations as

$$(5) \quad a_2 = \frac{3}{2}\alpha \quad \text{and} \quad \frac{2}{3}(3a_3 - 2a_2^2) = \beta, \quad \text{i.e.} \quad a_3 = \frac{1}{2}(\beta + 3\alpha^2)$$

for some $|\alpha| \leq 1$ and $|\beta| \leq 1$. Now we have to show that

$$(6) \quad \operatorname{Re}(s'_3(z)) = \operatorname{Re}(1 + 2a_2 z + 3a_3 z^2) > 0$$

in $|z| < 1/3$. Since the function $\operatorname{Re}(s'_3(z))$ is harmonic in $|z| \leq 1/3$, it is enough to prove (6) for $|z| = 1/3$. By considering a suitable rotation of f , it is enough to prove (6) for $z = 1/3$. Thus, it suffices to show that

$$(7) \quad \operatorname{Re} \left(1 + \frac{2}{3}a_2 + \frac{1}{3}a_3 \right) > 0.$$

By using the relations in (5) and the maximum principle, we see that the inequality (7) is equivalent to

$$(8) \quad \operatorname{Re} \left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\beta}{6} \right) > 0,$$

where $|\alpha| = 1$ and $|\beta| = 1$. If we take $\alpha = e^{i\theta}$ and $\beta = e^{i\phi}$ ($0 \leq \theta, \phi < 2\pi$), then in order to verify the inequality (8) it suffices to prove

$$\min_{\theta, \phi} T(\theta, \phi) > 0,$$

where

$$T(\theta, \phi) = 1 + \cos \theta + \frac{\cos 2\theta}{2} + \frac{\cos \phi}{6}$$

and θ, ϕ lies in $[0, 2\pi)$. Let

$$g(\theta) = 1 + \cos \theta + \frac{\cos 2\theta}{2}, \quad \theta \in [0, 2\pi).$$

Then

$$g'(\theta) = -\sin \theta(1 + 2 \cos \theta) \quad \text{and} \quad g''(\theta) = -[\cos \theta + 2 \cos 2\theta].$$

The points at which $g'(\theta) = 0$ are $\theta = 0, 2\pi/3, \pi$ and $4\pi/3$. But $g''(\theta)$ is positive for $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Hence

$$\min_{\theta} g(\theta) = g\left(\frac{2\pi}{3}\right) = g\left(\frac{4\pi}{3}\right) = \frac{1}{4}.$$

As the minimum value of $(\cos \phi)/6$ is $-1/6$, it follows that

$$\min_{\theta, \phi} T(\theta, \phi) = T\left(\frac{2\pi}{3}, \pi\right) = T\left(\frac{4\pi}{3}, \pi\right) = \frac{1}{12} > 0.$$

This proves the inequality (6) for $|z| < 1/3$.

Now let us consider the case $n \geq 4$. Let $f(z) = s_n(z) + \sigma_n(z)$, where $\sigma_n(z)$ is as given in Lemma E(c). Then

$$(9) \quad \operatorname{Re}(s'_n(z)) = \operatorname{Re}(f'(z) - \sigma'_n(z)) \geq \operatorname{Re}(f'(z)) - |\sigma'_n(z)|.$$

By maximum principle it is enough to prove that $\operatorname{Re}(s'_n(z)) > 0$ for $|z| = 1/3$. Now let us estimate the values of $\operatorname{Re}(f'(z))$ and $|\sigma'_n(z)|$ on $|z| = 1/3$.

As in the proof of Lemma E(b) in [12], we have the subordination relation for $f \in \mathcal{F}$,

$$(10) \quad f'(z) \prec \frac{1}{(1-z)^3}, \quad z \in \mathbb{D}.$$

We need to find the image of the circle $|z| = r$ under the transformation $w(z) = 1/(1-z)^3$. As the bilinear transformation $T(z) = 1/(1-z)$ maps the circle $|z| = r$ onto the circle

$$\left| T - \frac{1}{1-r^2} \right| = \frac{r}{1-r^2}, \quad \text{i.e., } T(z) = \frac{1 + re^{i\theta}}{1 - r^2},$$

a little computation shows that the image of the circle $|z| = r$ under the transformation $w = 1/(1 - z)^3$ is a closed curve described by

$$w = \frac{(1 + re^{i\theta})^3}{(1 - r^2)^3} = \frac{1 + r^3e^{3i\theta} + 3r^2e^{2i\theta} + 3re^{i\theta}}{(1 - r^2)^3}, \quad \theta \in [0, 2\pi).$$

From this relation, the substitution $r = 1/3$ gives that

$$\operatorname{Re} w = \left(\frac{9}{8}\right)^3 \left[1 + \cos \theta + \frac{\cos 2\theta}{3} + \frac{\cos 3\theta}{27}\right] = h(\theta) \text{ (say).}$$

If we write $h(\theta)$ in powers of $\cos \theta$, then we easily get

$$h(\theta) = \left(\frac{9}{8}\right)^3 \left[\frac{2}{3} + \frac{8}{9} \cos \theta + \frac{2}{3} \cos^2 \theta + \frac{4}{27} \cos^3 \theta\right].$$

If we let $x = \cos \theta$, then we can rewrite $h(\theta)$ in terms of x as

$$p(x) = \left(\frac{9}{8}\right)^3 \left[\frac{2}{3} + \frac{8}{9}x + \frac{2}{3}x^2 + \frac{4}{27}x^3\right],$$

where $-1 \leq x \leq 1$. In order to find the minimum value of $h(\theta)$ for $\theta \in [0, 2\pi)$, it is enough to find the minimum value of $p(x)$ for $x \in [-1, 1]$. A computation shows that

$$p'(x) = \frac{81(x+2)(x+1)}{128} \quad \text{and} \quad p''(x) = \frac{81(3+2x)}{128}.$$

In the interval $[-1, 1]$, $p'(x) = 0$ implies $x = -1$ is the only possibility. Also $p''(-1) > 0$ and so the minimum value of the function $p(x)$ in $[-1, 1]$ occurs at $x = -1$. The above discussion implies that

$$\min_{\theta} h(\theta) = h(\pi) = \frac{27}{64}.$$

Moreover, from the subordination relation (10), we deduce that

$$(11) \quad \min_{|z|=1/3} \operatorname{Re} (f'(z)) \geq \min_{|z|=1/3} \operatorname{Re} \left(\frac{1}{(1-z)^3} \right) = \frac{27}{64}.$$

Images of the disks $|z| < r$ for $r = 1/3, 1/2, 3/4, 4/5$, under the function $H(z) = 1/(1 - z)^3$ are drawn in Figures 1(a)-(d). From Lemma E(c), we have for $|z| = 1/3$

$$(12) \quad -|\sigma_n(z)| \geq \frac{-1}{8 \times 3^{n-1}} [2n^2 + 8n + 9] = k(n) \text{ (say).}$$

Now

$$k'(n) = \frac{-1}{8 \times 3^{n-1}} \left[\log \left(\frac{1}{3} \right) (2n^2 + 8n + 9) + 4n + 8 \right].$$

For $n \geq 4$, $k'(n) > 0$ and hence $k(n)$ is an increasing function of n . Thus for all $n \geq 4$, we have $k(n) \geq k(4) = -73/216$.

Finally, from the relations (9), (11) and (12) it follows that

$$\operatorname{Re} (s'_n(z)) > \frac{27}{64} - \frac{73}{216} = \frac{145}{1728} > 0 \text{ for all } n \geq 4.$$

The proof is complete.

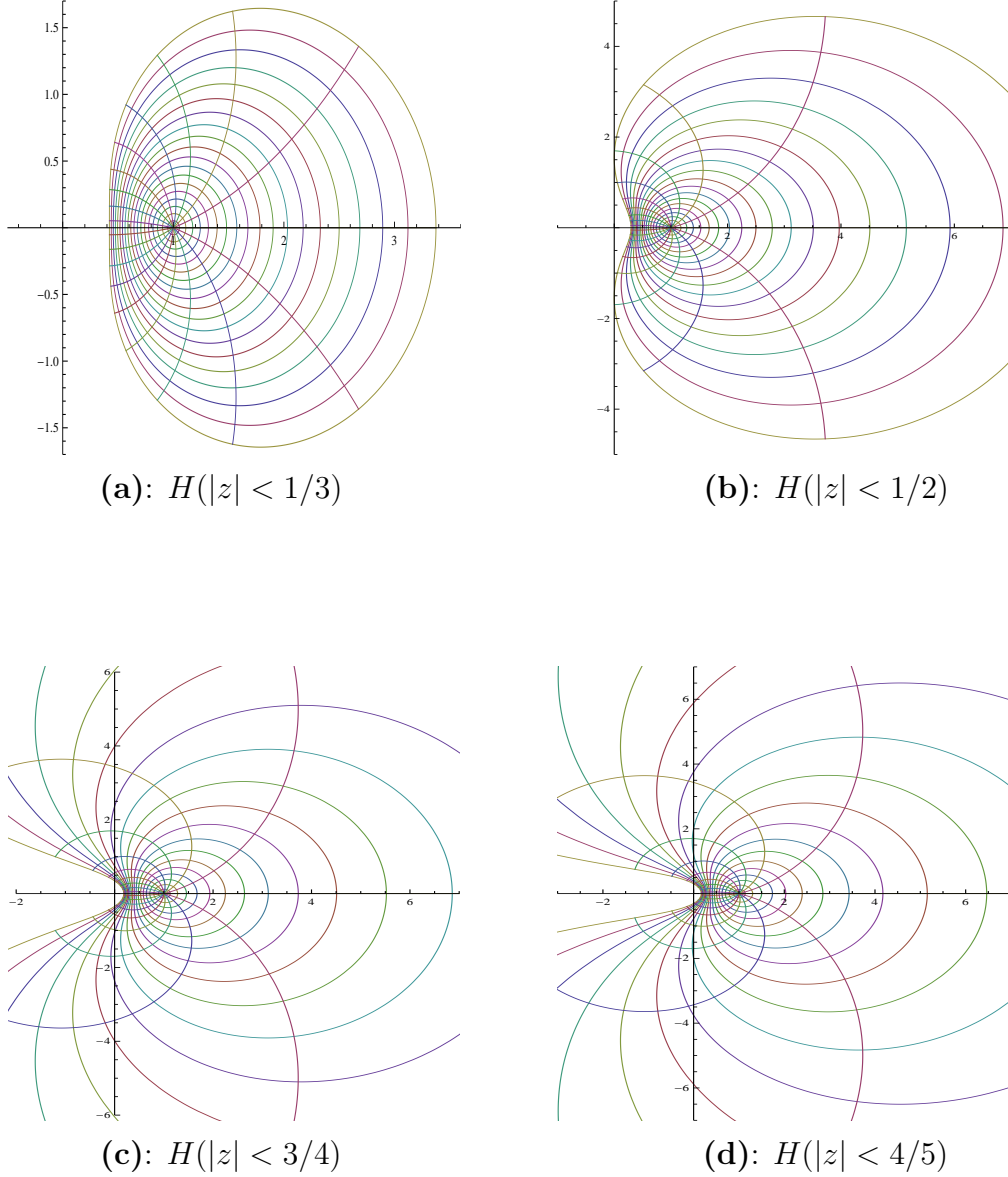


FIGURE 1. Images of the disks $|z| < r$ for $r = 1/3, 1/2, 3/4, 4/5$, under the function $H(z) = 1/(1-z)^3$.

We end the paper with the following conjecture.

Conjecture 2. *Every section $s_n(f)$ of $f \in \mathcal{F}$ is starlike in the disk $|z| < 1/3$.*

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